Fluid dynamical aspects

Eq^{n}s of horizontal motion

$$① \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} - ku$$

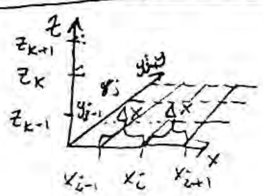
$$② \quad \underbrace{\frac{\partial v}{\partial t}}_{\text{local change}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}}_{\text{advective terms}} + \underbrace{fu}_{\text{coriolis}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial y}}_{\text{press. gradient}} \underbrace{-kv}_{\text{friction } (k>0)}$$

How to solve?

exact methods (possible only if we linearize eq^{n}s)
- non linear: unknown mult/div by another
  simplify adv. terms

approx. methods
* 1) finite difference method
2) spectral method } Galerkin
3) finite element method } approx.

Finite difference method: discretize space, time
discretize time into intervals, Δt ($t^n = n\Delta t$)



- use finite difference quotients (FDQ) to transform continuous derivatives

$$\text{e.g. } \left(\frac{\partial u}{\partial t}\right)^n_{ijk} \Rightarrow \frac{u(x_i, y_j, z_k, t^{n+1}) - u(x_i, y_j, z_k, t^{n-1})}{2\Delta t}$$

$$\Updownarrow$$

$$\frac{u^{n+1}_{ijk} - u^{n-1}_{ijk}}{2\Delta t}$$

Similarly, $u\left.\frac{\partial u}{\partial x}\right|^n_{ijk} \Rightarrow u^n_{ijk}\, \frac{u^n_{i+1jk} - u^n_{i-1jk}}{2\Delta x}$

notice $\lim_{\Delta t \to 0} \frac{u^{n+1}_{ijk} - u^{n-1}_{ijk}}{2\Delta t} \equiv \frac{\partial u}{\partial t}$; $\therefore$ this approx is consistent

many approximations are consistent! e.g. $\lim_{\Delta t \to 0} \frac{u^{n+1}_{ijk} - u^n_{ijk}}{\Delta t} \equiv \frac{\partial u}{\partial t}$

Which approximations are:

a) most accurate
b) " economical
c) should never be used

eq^n (1) in infinite difference form:

$$\frac{u_{ijk}^{n+1} - u_{ijk}^{n-1}}{2\Delta t} + u_{ijk}^{n} \frac{u_{i+1jk}^{n} - u_{i-1jk}^{n}}{2\Delta x} + v_{ijk}^{n} \frac{u_{ij+1k}^{n} - u_{ij-1k}^{n}}{2\Delta y} + W_{ijk}^{n} \frac{u_{ijk+1}^{n} - u_{ijk-1}^{n}}{2\Delta z}$$

$$- f_j\, v_{ijk}^{n} = -\frac{1}{\rho_{ijk}} \frac{p_{i+1jk}^{n} - p_{i-1jk}^{n}}{2\Delta x}$$

→ Order 20 operations per eq^n ~~per~~ grid point per time step

$\Delta t \sim 5$ min ⇒ 288 time steps per day!

Eq^{n}s: u, v, w, ρ, p ⇒ 5 variables minimum

grid points: latitude (ϕ), longitude (λ) coordinates — $\Delta\phi, \Delta\lambda \sim 2°$ ⇒ 100 lat × 200 lon.

10 vertical levels

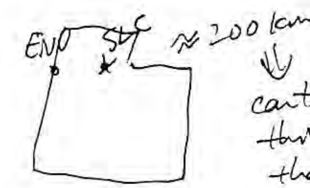
" 200,000 grid points!

⇒ $20 \times 288 \times 200{,}000 \times 2\times10^{5} \times 5$

⇒ 5×10^{9} operations for 1 day prog.

~ 1.5 hrs to run.

BUT,



⇓ can't resolve things smaller than @ these 2 pts.

for 10 km:

$20 \times 20 \times 5 \times 20$ ($\times 5\times10^{9}$ oper)
$\;\;x \;\;\; y \;\;\; z \;\;\; t$

⇒ 40,000 hrs
(or 40,000 processors)

Simplify ① + ② by neglecting the advection, pressure + friction terms.

1 → ③ $\frac{\partial u}{\partial t} - fv = 0$
2 → ④ $\frac{\partial v}{\partial t} - fu = 0$
} inertial oscillation

③ + i④: $\frac{\partial (u+iv)}{\partial t} - fv - fu = 0$

$\frac{\partial (u+iv)}{\partial t} + i\cdot i\, fv + i fu = 0$

$\frac{\partial (u+iv)}{\partial t} + i f(u+iv) = 0$

(5) $\frac{d(u+iv)}{dt} = -if(u+iv)$ ⇐ ODE w/const. coeffs.

all such equations have a solution of form

(6) $(u+iv) = Ae^{\gamma t}$ where A + γ are consts, t b d (to be determined)

A: substitute initial conditions $(u+iv)(t=0) = u_0$

$u_0 = Ae^{\gamma(0)} = A \Rightarrow A = u^0$

γ: substitute (6) into (5)

$\frac{d(u+iv)}{dt} = \cancel{A}\gamma \cancel{e^{\gamma t}} = -if\cancel{A}\cancel{e^{\gamma t}} \Rightarrow \gamma = -if$

Truncation Error (TE)

Suppose

$$\left.\begin{aligned} &\frac{\xi_{j+1}-\xi_j}{\Delta x} ; \left(\frac{\partial \xi}{\partial x}\right)_j \\ &\frac{\xi_{j+1}-\xi_{j-1}}{2\Delta x} ; \left(\frac{\partial \xi}{\partial x}\right)_j \end{aligned}\right\} \text{which FDQA is better?}$$

to answer: evaluate the TE.

to obtain TE, expand all variables appearing in the FDQA in a Taylor Series about the point of approximation

1) $\dfrac{\xi_{j+1}-\xi_j}{\Delta x} : \left(\dfrac{\partial \xi}{\partial x}\right)_j$

$$(a)\ \xi_{j+1} = \xi_j + \left(\frac{\partial \xi}{\partial x}\right)_j \Big(\overbrace{\frac{x_{j+1}-x_j}{1!}}^{\Delta x}\Big) + \underbrace{\left(\frac{\partial^2 \xi}{\partial x^2}\right)_j \frac{\overbrace{(x_{j+1}-x_j)^2}^{(\Delta x)^2}}{2!} + \dots}_{R\ \text{(Remainder)}}$$

$$(b)\ \xi_j = \xi_j + \underbrace{\cancel{\left(\frac{\partial \xi}{\partial x}\right)_j \frac{(x_j - x_j)}{1!}}}_{0}$$

$$|R| \le \text{max value of } \left|\frac{\partial^2 \xi}{\partial x^2}\frac{(\Delta x)^2}{2!}\right| \text{ on } (x_j, x_{j+1})$$

$$\frac{(a)-(b)}{\Delta x}$$

$$\underbrace{\frac{\xi_{j+1}-\xi_j}{\Delta x}}_{\text{approx}} = \underbrace{\left(\frac{d \xi}{\partial x}\right)_j}_{\text{deriv. being approx.}} + \underbrace{\left(\frac{d^2 \xi}{d x^2}\right)_j \frac{\Delta x}{2!} + \dots}_{\text{TE (error of approx.)}}$$

consider TSE w/ Remainder, we conclude (CBS) $|TE| \le \max \left|\dfrac{d^2 \xi}{\partial x^2}\dfrac{\Delta x}{2}\right|$ on (x_j, x_{j+1})

This is a 1st order approx (i.e. TE $\sim O(\Delta x^1)$) [Decreases proportionally to grid size]

e.g. (2) $\dfrac{\xi_{j+1}-\xi_{j-1}}{2\Delta x} : \left(\dfrac{\partial \xi}{\partial x}\right)_j$

$$(a)\ \xi_{j+1} = \xi_j + \left(\frac{d \xi}{\partial x}\right)_j \Delta x + \left(\frac{d^2 \xi}{\partial x^2}\right)_j \frac{\Delta x^2}{2!} + \dots$$

$$(b)\ \xi_{j-1} = \xi_j + \left(\frac{d \xi}{\partial x}\right)_j \underset{\uparrow\ \frac{x_{j-1}-x_j}{1!}}{(-\Delta x)} + \left(\frac{d^2 \xi}{\partial x^2}\right)_j \frac{\Delta x^2}{2!} + \dots$$

$$\frac{(a)-(b)}{2\Delta x} : \left.\frac{\xi_{j+1}-\xi_{j-1}}{2\Delta x}\right] \text{approx}$$

$$= \underbrace{\frac{d\xi}{dx_j}}_{\text{deriv.}} + \underbrace{\left(\frac{d^3 \xi}{\partial x^3}\right) \frac{\Delta x^2}{3!} + \dots}_{\text{TE}}$$

CBS, $|T.E| \le \max \left|\dfrac{d^3 \xi}{\partial x^3}\dfrac{\Delta x^2}{3!}\right|$ $\Rightarrow$ 2nd order, smaller TE (higher order)